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DIPLOMOVÁ PRÁCE

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Homogenita topologických struktur

Katedra matematické analýzy

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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

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Abstrakt: V předložené práci studujeme kompaktifikace, na které je možné spojitě rozšířit homeomorfismy základního prostoru. Takové nazýváme H-kompaktifikacemi. Charakterizujeme je pomocí několika ekvivalentních podmínek a dokážeme, že H-kompaktifikace daného prostoru tvoří úplný horní polosvaz, který je úplným svazem v případě, že výchozí prostor je lokálně kompaktní. Dále popisujeme všechny H-kompaktifikace diskrétních prostorů a spočetných lokálně kompaktních prostorů. Ukazujeme, že jediné H-kompaktifikace Euklidovských prostorů dimenze alespoň dvě jsou jednobodová a Čechova-Stonova kompaktifikace. Následně obdržíme, že existuje přesně 11 H-kompaktifikací spočetné sumy Euklidovských prostorů dimenze alespoň dvě a přesně 26 H-kompaktifikací spočetné sumy reálných přímek. Ty jsou všechny popsány a je dán Hasseho diagram svazu, který tvoří.

Klíčová slova: kompaktifikace, Euklidovský prostor, homogenita, homogenní zobrazení

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Abstract: In the present work we study those compactifications such that every autohomeomorphism of the base space can be continuously extended over the compactification. These are called H-compactifications. We characterize them by several equivalent conditions and we prove that H-compactifications of a given space form a complete upper semilattice which is a complete lattice when the given space is supposed to be locally compact. Next, we describe all H-compactifications of discrete spaces as well as of countable locally compact spaces. It is shown that the only H-compactifications of Euclidean spaces of dimension at least two are one-point compactification and the Čech-Stone compactification. Further we get that there are exactly 11 H-compactifications of a countable sum of Euclidean spaces of dimension at least two and that there are exactly 26 H-compactifications of a countable sum of real lines. These are all described and a Hasse diagram of a lattice they form is given.

Keywords: compactification, Euclidean space, homogeneity, homogeneous mapping

Preface

This thesis is focused on a notion of an H-compactification. It is a compactification such that all autohomeomorphisms of the base space can be continuously extended over the compactification. We note that the notion was partially studied by several authors in the papers [Smi94] where it is called *equivariant extension*, in [dGM60] it is called *G-compactification*, or in [vD79] where the author used a name *topological compactification*. The last one is probably the most suitable name since it express that the compactification is defined only with respect to the topology of the base space and does not depend on the concrete representation of the space. However, this collocation of words is too long so that it could be used frequently.

It is already known that Sorgenfrey line as well as spaces of rational and irrational numbers admit only one H-compactification and that there are exactly three H-compactifications of the real line [vD79]. In the same paper it is noted that there are at least eleven H-compactifications of a countable sum of real lines. We proceed in this work when studying Euclidean spaces of heigher dimension and their countable sums. In the last chapter we continue in a research of a paper [dGM60], where it is proved that under some conditions a metric space possess a metric H-compactification.

Next, we describe several conventions which may be helpful for better reading of this thesis and for elimination of misunderstandings.

Letters i and j are used mainly as indeces. Whenever $e\colon X \to Y$ and $f\colon Y \to Z$ are mappings we denote by $fe\colon X \to Z$ their composition. The symbol 'id_X' or just 'id' means the identity function on the set X. For any collection of mappings $f_i\colon X \to Y_i$ for $i \in I$ a diagonal mapping $\Delta_{i\in I} f_i\colon X \to \prod_{i\in I} Y_i$ is given by $\Delta f_i(x) = \big(f_i(x)\big)_{i\in I}$. When dealing with ordinal numbers we usually use Greek letters δ or ϵ. For cardinal numbers we reserve letters κ, λ and μ. Cardinal successor of κ is denoted by κ^+. The least infinite ordinal is labeled by the letter ω and its elements by k, l, m, n. It turns out that using a shortcut $\{0, \dots, n-1\} = n$ is efficient.

By letters X, Y and Z we usually mean topological spaces and we denote by E, F and G, H, U, V their closed and open subsets respectively. Closure and interior of a set A in X are denoted by $\overline{A}^X$ and $\mathrm{int}_X A$, or just by $\overline{A}$ and int A. The symbol $\mathcal{H}(X)$ means the group of all autohomeomorphisms of the space X. The fact that two spaces X and Y are homeomorphic is symbolically abbreviated by $X \cong Y$. Letters g and h are reserved for homeomorphisms. For an equivalence $\sim$ on X we write $X/_\sim$ for the corresponding quotient space. When this quotient arise simply by shrinking a set $A \subseteq X$ we write just X/A. For the set of all continuous mappings from X to Y we use $\mathcal{C}(X, Y)$ and its elements are denoted by e and f. Then $\mathcal{C}(X)$ stands for all real-valued continuous functions and $\mathcal{C}^*(X)$ for bounded real-valued continuous functions.

A compact space is automatically supposed to be Hausdorff. By γX and δX we denote some compactifications of a Tychonoff space X. We accept commonly used notation for the embedding $\gamma\colon X \to \gamma X$. We hope, it is always clear, what the symbol 'γ' means. Alexandroff one-point compactification of a non-compact locally compact space X is signed by αX. By βX we denote Čech-Stone compactification. For a given continuous mapping $f\colon X \to Y$ of Tychonoff spaces we denote by $\beta f\colon \beta X \to \beta Y$ the only continuous extension of f. For simplification we define X^* to be the Čech-Stone remainder $\beta X \setminus X$.

When working with metric spaces, we use $\operatorname{dist}(A, B)$ for the distance of two sets and $\operatorname{diam} A$ to denote diameter of a set. Note that $\operatorname{dist}(A, B)$ is by definition equal to infinity whenever A or B is empty and $\operatorname{diam} A$ equals zero whenever A is empty. The bold letters $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for integers, rationals and real numbers respectively. The symbol $\mathbb{S}^n$ is used for the n-dimensional sphere, not for a power. Thus $\mathbb{S}^1 = \mathbb{S}$ is just a circle.

For a given space X and a given point $x \in X$ a set $\{h(x)\colon h \in \mathcal{H}(X)\}$ is called *orbit* of the point x. Remind that a space X is said to be *homogeneous* if for every pair of points $x, y \in X$ there is a homeomorphism $h \in \mathcal{H}(X)$ for which $h(x) = y$. In other words there is at most one orbit. On the other hand a space X is called *rigid* whenever $\mathcal{H}(X)$ contains only identity. This means every single point in X forms an orbit. Moreover a space X is said to be *bihomogeneous* if for every pair of points $x, y \in X$ there exists a homeomorphism $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $h(y) = x$.

For all common notions and topological ideas used in this thesis we refer to [Eng89]. Note that some of them are used implicitly. Lot of facts about compactifications can be found in [Cha76].

Chapter 1

Homogeneous mappings

In this chapter we are going to define a strong and helpful notion of a homogeneous mapping. We prove a statement concerning relationship with rigidity and homogeneity. Further we establish some simple but useful facts about homogeneous mappings.

Definition 1.1. We say that a continuous mapping $f\colon X \to Y$ is *homogeneous* if for every $h \in \mathcal{H}(X)$ there exists $g \in \mathcal{H}(Y)$ such that $fh = gf$.

Let us emphasize that currently defined notion depends essentially on the codomain of the mapping f. It is easy to verify that identity as well as constant mappings are always homogeneous and that a composition of two homogeneous mappings is again homogeneous. Neither products nor sums of homogeneous mappings are in general homogeneous. To see this consider a pair of inclusion mappings $\omega \to \omega+1$ and $\omega \to \omega+1$ sum and product of which is not homogeneous. On the other hand we prove that diagonal mapping of homogeneous mappings is homogeneous. As can be easily shown a homogeneous mapping defined on a bihomogeneous space is either one-to-one or constant.

Proposition 1.2. *There exist a T_0 (resp. Tychonoff) space Y such that for every T_0 (resp. Tychonoff) space the following are equivalent.*

(i) *A space X is rigid.*

(ii) *Every continuous mapping $f\colon X \to Y$ is homogeneous.*

Proof. Clearly every continuous mapping of a rigid space into arbitrary space is homogeneous. Thus it remains to prove $(ii) \to (i)$. We define Y to be the Sierpiǹski space $\big(2, \{2, 1, \emptyset\}\big)$ (resp. closed interval [0,2]). Suppose for contradiction that there exists a homeomorphism $h \in \mathcal{H}(X)$ and two distinct points $x, y \in X$ for which $h(x) = y$. Since the space X is T_0

there exists an open set G containing the point x and not containing y (or vice versa – but the same argumentation can be used then). Define a mapping $f\colon X \to Y$ by the rule

$$f(z) = \begin{cases} 0, & z \in G \\ 1, & z \notin G \end{cases}$$

(resp. to be any continuous mapping into $[0,2]$ for which $f(x) = 0$ and $f\restriction_{X\setminus G} = 1$). Clearly the mapping f is not homogeneous because there is no homeomorphism of Y sending the point $f(x)$ to $f(y)$. That is a contradiction. $\square$

Remark 1.3. The assumption for a space X in Proposition 1.2 to satisfy at least any separating axioms is essential since one can realize that indiscrete space X containing two points is not rigid but every continuous mapping $f\colon X \to Y$ is homogeneous for an arbitrary space Y.

Moreover the assumption for X and Y being T_0 (resp. Tychonoff) in Proposition 1.2 cannot be replaced neither by T_1, T_2 nor T_3. Thus suppose for contradiction there is a T_1 space Y such that for every T_3 space X we have that X is rigid if and only if every continuous mapping $f\colon X \to Y$ is homogeneous. By Herrlich result noted in [Eng89, p. 119] there exists a T_3 space Z with at least two points such that $\mathcal{C}(Z,Y)$ consists of constant mappings only. If we put $X = Z \times Z$ we get a non-rigid T_3 space such that every continuous mapping $f\colon X \to Y$ is constant, hence homogeneous. This contradicts our assumptions.

The following proposition give reason for the notion of a homogeneous mapping.

Proposition 1.4. *An image of a homogeneous space under a homogeneous mapping is again homogeneous.*

Proof. Let $f\colon X \to Y$ be a homogeneous mapping of a homogeneous space X onto Y. Take any two points y, y' in Y. There exists $x, x' \in X$ for which $f(x) = y$ and $f(x') = y'$. Since X is homogeneous there exists a homeomorphism $h \in \mathcal{H}(X)$ such that $h(x) = x'$. By homogeneity of the mapping f there is a homeomorphism $g \in \mathcal{H}(Y)$ such that $fh = gf$. Thus $g(y) = gf(x) = fh(x) = f(x') = y'$. $\square$

Lemma 1.5. *Diagonal mapping $f = \Delta f_i\colon X \to \prod Y_i$ is homogeneous provided that all the mappings $f_i\colon X \to Y_i$ are homogeneous.*

Proof. Take arbitrary $h \in \mathcal{H}(X)$. For each i there exists $g_i \in \mathcal{H}(Y_i)$ such that $f_i h = g_i f_i$. Now $g = \prod g_i \in \mathcal{H}\big(\prod Y_i\big)$ satisfies $fh(x)_i = f_i h(x) = g_i f_i(x)$ for every i and every $x \in X$. Hence $fh = gf$ and so the diagonal mapping is homogeneous. $\square$

Lemma 1.6. *Let $f\colon X \to Y$ be a homogeneous mapping. Than its corestrictions $f\colon X \to f(X)$ and $f\colon X \to \overline{f(X)}$ are homogeneous too.*

Proof. For any $h \in \mathcal{H}(X)$ there exists $g \in \mathcal{H}(Y)$ such that $fh = gf$. From this we derive that $g\restriction_{f(X)}$ is a homeomorphism of $f(X)$ onto $f(X)$. Thus $f\colon X \to f(X)$ is homogeneous.

Since the mappings g and g^{-1} are continuous and $gf(X) = f(X)$ we get that $g\big(\overline{f(X)}\big) = \overline{f(X)}$. Thus $g\restriction_{\overline{f(X)}}$ is a homeomorphism onto $\overline{f(X)}$. Thus the mapping $f\colon X \to \overline{f(X)}$ is homogeneous too. $\square$

Chapter 2

Lattice structure

In this chapter we define first a notion of an H-compactification. Further we prove that H-compactifications of one fixed space behave well in the sense that their supremum and infimum (if it exists) is again an H-compactification. Moreover we give a characterization of an H-compactification using bounded continuous functions which are extendable over the compactification.

Definition 2.1. A compactification γX of a Tychonoff space X is said to be an *H-compactification* provided that the inclusion mapping $\gamma\colon X \to \gamma X$ is homogeneous.

This means every homeomorphism $h \in \mathcal{H}(X)$ can be extended to a homeomorphism $g \in \mathcal{H}(\gamma X)$. It can be equivalently reformulated to the condition that every $h \in \mathcal{H}(X)$ is continuously extendable over γX. Čech-Stone compactification βX is an H-compactification because every $h \in \mathcal{H}(X)$ can be extended continuously to a mapping $\beta h\colon \beta X \to \beta X$. One-point compactification $\alpha X = X \cup \{\infty\}$ is an H-compactification for non-compact locally compact space X too. One can namely take for every $h \in \mathcal{H}(X)$ the only bijection $g\colon \alpha X \to \alpha X$ extending h and verify that g is a homeomorphism because open neighbourhoods of the point ∞ are exactly complements in αX of compact subsets of X.

Informally, a compactification is an H-compactification if and only if it can be defined only in dependence on topological properties of the given space. It is easy to verify that every reflection of a full subcategory of topological spaces into its subcategory consists of homogeneous mappings. Thus we can derive that e.g. Banaschewski compactification (the biggest zero-dimensional compactification) of a zero-dimensional space is always an H-compactification.

Recall that for a Tychonoff space X we can consider a set of all (up to equivalence) compactifications of X with natural order which assigns to this set a complete upper semilattice structure. When the space X is taken to be locally compact we get even a complete lattice. For more details see [Cha76, p. 16].

Proposition 2.2. *The semilattice of all H-compactifications of a Tychonoff space X is a complete subsemilattice of the semilattice of all compactifications of X.*

Proof. For H-compactifications $\gamma_i X$ we denote by $\gamma_i\colon X \to \gamma_i X$ the inclusion mappings. Least upper bound of these compactifications is given by $\overline{\gamma(X)}$ where $\gamma = \Delta\gamma_i$. Now this new compactification is an H-compactification because the mapping γ and its corestriction $\gamma\colon X \to \overline{\gamma(X)}$ are homogeneous by Lemmas 1.5 and 1.6. $\square$

Theorem 2.3. *Let γX be a compactification of a Tychonoff space X, let $\mathcal{U}$ be the only uniformity on γX and let $\gamma X = \beta X/_\sim$. Then the following conditions are equivalent.*

- *(i)* *γX is an H-compactification.*
- *(ii)* *For every $h \in \mathcal{H}(X)$ and $x, y \in \beta X$ such that $x \sim y$ we have that $\beta h(x) \sim \beta h(y)$.*
- *(iii)* *Every homeomorphism $h \in \mathcal{H}(X)$ is uniformly continuous with respect to the uniformity $\mathcal{U}$.*
- *(iv)* *For every $f \in \mathcal{C}^*(X)$ which is continuously extendable over γX and for every $h \in \mathcal{H}(X)$ the function fh can be continuously extended over γX.*

Proof. *(i)* $\to$ *(ii)* Take $h \in \mathcal{H}(X)$ and $x, y \in \beta X$. Denote by $\varphi\colon \beta X \to \gamma X$ the only extension of identity on X. Note that $x \sim y$ iff $\varphi(x) = \varphi(y)$. By *(i)* there is a homeomorphism $g \in \mathcal{H}(\gamma X)$ extending h from which we get that $\varphi\beta h = g\varphi$ since left and right side of this equality are the same on a dense subset of βX. Thus if $\varphi(x) = \varphi(y)$ we get that $\varphi\beta h(x) = g\varphi(x) = g\varphi(y) = \varphi\beta h(y)$ and hence $\beta h(x) \sim \beta h(y)$.

(ii) $\to$ *(iii)* For any $h \in \mathcal{H}(X)$ define $g \in \mathcal{H}(\beta X/_\sim)$ by a condition $g\big([x]_\sim\big) = \big[\beta h(x)\big]_\sim$. The mapping g is uniformly continuous and extends h. Thus h is uniformly continuous too.

(iii) $\to$ *(iv)* Take any $h \in \mathcal{H}(X)$ and $f \in \mathcal{C}^*(X)$ which is continuously extendable into a mapping $\bar{f}\colon \gamma X \to \mathbb{R}$. Since h is uniformly continuous by *(iii)* and $\bar{f}$ is uniformly continuous too, the composition fh is uniformly continuous. Thus by Theorem 8.3.10 from [Eng89, p. 447] we get that there is a continuous extension of the mapping fh over γX.

(iv) $\to$ *(i)* Let $h \in \mathcal{H}(X)$ be given. We want to show that h is continuously extendable into a mapping $\gamma X \to \gamma X$. Let us remind Theorem 3.2.1 from [Eng89, p. 136].

A continuous mapping $h\colon X \to Z$, where X is a dense subset of a space Y and Z is compact, can be continuously extended over X if and only if for every pair E, F of disjoint closed subsets of Z the preimages $h^{-1}(E)$ and $h^{-1}(F)$ have disjoint closures in Y.

In this situation we set $Y = \gamma X = Z$. In order to verify the condition let us fix two disjoint closed subsets E and F of γX. There exists a continuous function $f \in \mathcal{C}(\gamma X)$ such that $E \subseteq f^{-1}(0)$ and $F \subseteq f^{-1}(1)$. Because of the condition (iv) there exists a continuous function $e \in \mathcal{C}(\gamma X)$ such that $fh = e\restriction_X$. Finally we have

$$\overline{h^{-1}(E)} \cap \overline{h^{-1}(F)} \subseteq \overline{(fh)^{-1}(0)} \cap \overline{(fh)^{-1}(1)} \subseteq e^{-1}(0) \cap e^{-1}(1) = \emptyset.$$

Thus by the cited theorem we are done. $\square$

Using the previous theorem we get the following result.

Proposition 2.4. *The lattice of all H-compactifications of a locally compact space X is a complete sublattice of the lattice of all compactifications of X.*

Proof. For given H-compactifications $\gamma_i X$ denote by C_i the sets of all continuous functions from $\mathcal{C}^*(X)$ that are extendable over $\gamma_i X$. Let $C = \bigcap C_i$. The greatest lower bound of compactifications $\gamma_i X$ is given by $\overline{\gamma(X)}$ where $\gamma\colon X \to \mathbb{R}^C$ is defined by the equality $f(x)_e = e(x)$. We are going to verify the fourth condition from the characterization of H-compactifications in Theorem 2.3. Let $h \in \mathcal{H}(X)$ and $f \in \mathcal{C}^*(X)$ be continuously extendable over $\overline{\gamma(X)}$. Then $f \in C$. Consequently $fh \in C_i$ for every i because $\gamma_i X$ are H-compactifications. Hence $fh \in \bigcap C_i = C$ is continuously extendable over $\overline{\gamma(X)}$. $\square$

Chapter 3

Countable, discrete and Euclidean spaces

How do look like all H-compactifications of a given topological space? This question seems to be hard to answer in some cases. For example when a rigid space is selected then every compactification is an H-compactification and there can be a huge number of them (e.g. when a dense rigid subset of the real line is taken there exist at least continuum many of them). However, we describe all H-compactifications of discrete, Euclidean and countable locally compact spaces. This is possible since these spaces possess a lot of homeomorphisms. Finally we prove that a sum of countable many copies of the real line admits exactly 26 H-compactifications.

3.1 Discrete spaces

Let us settle two lemmas that will be used when proving Theorem 3.3.

Lemma 3.1. *Let X be a Tychonoff space and γX and δX be two compactifications such that δX is an H-compactification of X and there exists a continuous mapping $f\colon \delta X \to \gamma X$ extending identity on X. If f is homogeneous mapping then γX is an H-compactification of X.*

Proof. The inclusion $\gamma\colon X \to \gamma X$ is just a composition of two homogeneous mappings. $\square$

Note that the opposite implication in the previous lemma is not true in general. Just consider a space $X = \beta\omega \oplus \omega$ and continuous mapping $f\colon \beta X \to \alpha X$ which extends identity on X. Note that $\beta X = \beta\omega \oplus \beta\omega$ is an H-compactification of X and observe that f is not a homogeneous mapping.

Lemma 3.2. *Let $\sim$ be an equivalence relation on a space X. Suppose moreover that equivalence classes are either singletons or unions of some orbits. Then the quotient mapping $q\colon X \to X/_\sim$ is homogeneous.*

Proof. Fix arbitrary $h \in \mathcal{H}(X)$. And define $g\colon X/_\sim \to X/_\sim$ by the rule $g\big([x]\big) = \big[h(x)\big]$, where $[x]$ denotes the equivalence class containing x. This definition is correct because equivalence classes are either singletons or unions of some orbits. The mapping g is a bijection satisfying $qh = gq$. For an open set $G \subseteq X/_\sim$ we can see that $H = h^{-1}q^{-1}(G)$ is open. If $x \in H$ then $[x] \subseteq H$. Hence $q(H)$ is open. The equality $q(H) = g^{-1}(G)$ implies that g is continuous. Similarly g^{-1} is continuous too and consequently g is required homeomorphism. □

Theorem 3.3. *The only H-compactifications of a discrete space D of cardinality $\kappa \geq \omega$ are $\beta D/F_\lambda$ where $\omega \leq \lambda \leq \kappa^+$ and*

$$F_\lambda = \beta D \setminus \bigcup \left\{\overline{A}\colon A \subseteq D, |A| < \lambda\right\}.$$

Proof. Remind that the system $\left\{\overline{A}\colon A \subseteq D\right\}$ is clopen bases of βD. Hence the sets F_λ are closed and $\beta D/F_\lambda$ are compactifications of D. These are all H-compactifications by Lemma 3.1 because quotient mappings $q_\lambda\colon \beta D \to \beta D/F_\lambda$ are homogeneous by Lemma 3.2. Note that $F_\omega = \beta D \setminus D$, $F_{\kappa^+} = \emptyset$ and hence $\beta D/F_\omega = \alpha D$ and $\beta D/F_{\kappa^+} = \beta D$.

Let $f\colon \beta D \to \gamma D$ be a continuous mapping onto γD extending the identity on D. And suppose the compactification γD is not equivalent to any $\beta D/F_\lambda$. This means that for every $\omega \leq \lambda \leq \kappa^+$ the inequality $|f(F_\lambda)| \geq 2$ holds or the restriction $f\restriction_{\beta D\setminus F_\lambda}$ is not one-to-one. Now let λ be the least cardinal such that the restriction $f\restriction_{\beta D\setminus F_\lambda}$ is not one-to-one. Clearly $\omega < \lambda \leq \kappa^+$. Because of minimality of λ we derive that $\lambda = \mu^+$ is cardinal successor and $|f(F_\mu)| \geq 2$.

This proof is completed by the following lemma. □

Lemma 3.4. *Let D be a discrete space of cardinality $\kappa \geq \omega$. Suppose $f\colon \beta D \to \gamma D$ is the only continuous extension of identity on D onto some compactification γD. If there exists $\omega \leq \mu \leq \kappa$ for which $|f(F_\mu)| \geq 2$ and $f\restriction_{\beta D\setminus F_{\mu^+}}$ is not one-to-one then γD is not an H-compactification of D.*

Proof. There exist two points $u, v \in F_\mu$ such that $f(u) \neq f(v)$ and two distinct points $x, y \in \beta D \setminus F_{\mu^+}$ such that $f(x) = f(y)$. Let U, V be open neighbourhoods of u and v respectively satisfying $f(U) \cap f(V) = \emptyset$. We can find $A, B \subseteq D$ such that $u \in \overline{A} \subseteq U$ and $v \in \overline{B} \subseteq V$. Note that $|A|, |B| \geq \mu$. From the fact that $x, y \notin F_{\mu^+}$ we can derive the existence of disjoint subsets $M, N \subseteq D$ of cardinalities less than μ^+ such that $x \in \overline{M}$, $y \in \overline{N}$. We can also require $|D \setminus (M \cup N)| = \kappa$.

It is easy to realize that there exists a bijection $b \in \mathcal{H}(D)$ so that $b(M) \subseteq A$ and $b(N) \subseteq B$. Let $h \in \mathcal{H}(\beta D)$ be the only continuous extension of b. Note that $h(x) \in \overline{A}$ and $h(y) \in \overline{B}$. Suppose for contradiction that γD is an H-compactification and find $g \in \mathcal{H}(\gamma D)$ extending b. Two continuous mappings fh and gf are equal on D and because D is dense in βD we derive $fh = gf$. On the other hand we have $fh(x) \in f(U)$ and $fh(y) \in f(V)$ and $gf(x) = gf(y)$. But $f(U)$ and $f(V)$ were disjoint hence $fh(x) \neq fh(y)$ and this is contradiction with the equality $fh = gf$. □

Corollary 3.5. *The only H-compactifications of ω are $\alpha\omega$ and $\beta\omega$.*

3.2 Countable metrizable spaces

As mentioned in a paper of de Groot and McDowell [dGM60], 'it is of interest to ask for conditions under which every autohomeomorphism of a given space can be extended to a suitable metric compactification'. In this section we study such a problem in the class of countable metrizable spaces. Let us recall frequently used fact that Alexandroff one-point compactification of a non-compact separable locally compact metrizable space is always metrizable.

Proposition 3.6. *A countable metrizable space admits a metrizable H-compactification if and only if it is locally compact. In this case the only metrizable H-compactification is the one-point compactification.*

Proof. Suppose that X is a countable metrizable space and γX an arbitrary metrizable H-compactification. Denote by Y the set of all isolated points of X. We claim that the set Y is dense in X. Suppose this is not true. Hence we get an open non-empty set $X \setminus \overline{Y}$ no points of which are isolated. We can find a non-empty clopen set $Q \subseteq X \setminus Y$. Since Q is a non-empty countable metrizable space without isolated points we obtain that it is homeomorphic to rational numbers $\mathbb{Q}$ (see [vE86, p. 17]). By a result of van Douwen [vD79] there exists exactly one H-compactification of $\mathbb{Q}$, namely $\beta\mathbb{Q}$. The inclusion $Q \to X$ is homogeneous which implies that the closure of Q in γX is homeomorphic to βQ. This is a contradiction with the assumption that γX is metrizable. So the claim is proved.

For arbitrary pair of points $x, y \in \gamma X \setminus X$ we can fix two sequences $x_n, y_m \in Y$ converging to x and y respectively because γX is metrizable and isolated points of X are dense in γX . We can assume moreover that all the points x_n, y_m are distinct. We are allowed to define a homeomorphism $h \in \mathcal{H}(X)$ for which

$$h(z) = \begin{cases} x_n, & \text{if } z = y_n \text{ and } n \text{ is odd,} \\ y_n, & \text{if } z = x_n \text{ and } n \text{ is odd,} \\ z, & \text{otherwise,} \end{cases}$$

because the sets $\{x_n\colon n \in \omega\}$ and $\{y_m\colon m \in \omega\}$ are clopen and discrete. There is an extension $g \in \mathcal{H}(\gamma X)$ of h for which necessarily $x = h(x) = y$. Thus $\gamma X \setminus X$ contains at most one point. Therefore X is locally compact.

Since one-point compactification of a locally compact separable metrizable space is again metrizable, we are done. $\square$

In the sequel, it is of particular interest that every countable locally compact space is metrizable. To see this remind that a network weight is the same as weight for compact spaces [Eng89, p. 127] and thus any countable locally compact space is first countable. Moreover by the Urysohn metrization theorem [Eng89, p. 260] we get that such a space is metrizable because it is second countable and regular.

Let us recall some generally known facts about *Cantor-Bendixson derivative* and *Cantor-Bendixson rank* that can be found in [HNV04, p. 350]. These will be helpful for a description of all H-compactifications of any countable locally compact space.

For any topological space X we can define its *derivative* X' to be the subspace of X which consists of all non-isolated points of X. Clearly, the space X' can contain some isolated points, so it make sense to repeat the derivative. This can be done transfinitely for all ordinals. For an isolated ordinal δ we let X^δ to be the derivation of the predecessor and for a limit ordinal δ we define X^δ to be the intersection of the preceding chain. Thus we get a sequence indexed by ordinal numbers. Note that all the sets X^δ are closed in X. Since this process has to stop we can define so called *Cantor-Bendixson rank* of X to be the smallest ordinal δ for which $X^\delta = X^{\delta+1}$. It will be simply denoted by $\operatorname{rank} X$. The space X is called *scattered* provided that the sequence X^δ is eventually empty set. Note that a countable locally compact space is always scattered.

For any space X define $\operatorname{top} X = X^\delta$ if $\operatorname{rank} X = \delta + 1$ and $\operatorname{top} X = \emptyset$ otherwise. Notice that for any scattered compact space its Cantor-Bendixson rank cannot be a limit ordinal. This follows easily from the fact that intersection of a chain of non-empty compact subspaces is non-empty. Thus we get $\operatorname{rank} X = \delta + 1$ for some ordinal δ. Since X^δ is a compact space, whose derivative is empty set, we get that $\operatorname{top} X = X^\delta$ is finite.

For our next work it is helpfull to define an ordinal $\operatorname{rank}_c X$ associated to every scattered space X. It is the least ordinal number δ for which X^δ is compact. Note that $\operatorname{rank}_c X \leq \operatorname{rank} X$.

Below we give a series of statements most of which are needed to a proof of Theorem 3.11.

Lemma 3.7 (Mazurkiewicz – Sierpiński [MS20])**.** *Let K and L be countable and compact spaces. Then K is homeomorphic to L if and only if rank K = rank L and top $K \cong$ top L.*

Corollary 3.8. *The orbits of a countable locally compact space X are precisely the sets $X^\delta \setminus X^{\delta+1}$ where $\delta <$ rank X. Moreover for any points $x, y \in X^\delta \setminus X^{\delta+1}$ there is a homeomorphism $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $h\restriction_{X^{\delta+1}}$ is equal to identity.*

Proof. Clearly the sets $X^\delta \setminus X^{\delta+1}$ are preserved by all homeomorphisms since Cantor-Bendixson derivative is a topological notion. On the other hand suppose we have two distinct points $x, y \in X^\delta \setminus X^{\delta+1}$. There exist disjoint compact clopen neighbourhoods P and Q of x and y respectively for which $P \cap X^\delta = \{x\}$ and $Q \cap X^\delta = \{y\}$. The sets P and Q are homeomorphic by Lemma 3.7 and since they are clopen we can extend this homeomorphism into a homeomorphism of the whole space by identity. $\square$

Proposition 3.9. *Let X and Y be two countable locally compact spaces. Then X is homeomorphic to Y if and only if rank $X =$ rank Y, rank$_c X =$ rank$_c Y$ and top $X \cong$ top Y.*

Proof. The direct implication is clear. If one of the two spaces is compact the second one is too, since $\text{rank}_c X = \text{rank}_c Y$, and by Lemma 3.7 we get desired result. Thus suppose they are both non-compact and consider their one-point compactifications $\alpha X = X \cup \{\infty_x\}$ and $\alpha Y = Y \cup \{\infty_y\}$. There are three possibilities.

- rank $X = \text{rank}_c X$. This implies that rank $\alpha X =$ rank $X + 1 =$ rank $Y + 1 =$ rank αY and top $\alpha X = \{\infty_x\} \cong \{\infty_y\} =$ top αY.
- rank $X = \text{rank}_c X + 1$. In this case rank $\alpha X =$ rank $X =$ rank $Y =$ rank αY and top $\alpha X =$ top $X \cup \{\infty_x\} \cong$ top $Y \cup \{\infty_y\} =$ top αY.
- rank $X > \text{rank}_c X + 1$. Then rank $\alpha X =$ rank $X =$ rank $Y =$ rank αY and top $\alpha X =$ top $X \cong$ top $Y =$ top αY.

In all cases the assumptions of Lemma 3.7 are satisfied hence there exists a homeomorphism $h\colon \alpha X \to \alpha Y$. It is not always the case that $h(\infty_x) = \infty_y$ so we need to find a homeomorphism $g \in \mathcal{H}(\alpha Y)$ for which $g\big(h(\infty_x)\big) = \infty_y$. However its existence is a consequence of the corollary 3.8 for $\delta = \text{rank}_c X$ since $\infty_x \in (\alpha X)^\delta \setminus (\alpha X)^{\delta+1}$ and thus $h(\infty_x), \infty_y \in (\alpha Y)^\delta \setminus (\alpha Y)^{\delta+1}$.

Finally $gh\restriction_X$ is the required homeomorphism. $\square$

The following proposition, although more or less unrelated to H-compactifications, is an interesting result.

Proposition 3.10. *Let X be a countable locally compact space and δ arbitrary ordinal number. Then the inclusion mapping $X^\delta \to X$ is homogeneous.*

Proof. Let $h \in \mathcal{H}(X^\delta)$ be a given homeomorphism we want to extend over X. Suppose first that the space X is compact. Assume ρ is a compatible metric. The notions such as diameter and distance are derived from this metric.

We claim that there exists a disjoint system C_x for $x \in X^\delta \setminus X^{\delta+1}$ of clopen sets satisfying $\bigcup C_x = X \setminus X^{\delta+1}$, $C_x \cap X^\delta = \{x\}$ and diam C_x tends to zero.

Since the space X is compact and zero-dimensional we are able to find a pairwise disjoint non-empty clopen sets O_n such that $\bigcup O_n = X \setminus X^\delta$, diam O_n tends to zero and $\operatorname{dist}(O_n, X^\delta)$ too. Now, for every n we can find $y_n \in X^\delta$ whose distance from O_n is the least. Since $X^\delta \setminus X^{\delta+1}$ is dense in X^δ there exists a point $x_n \in X^\delta \setminus X^{\delta+1}$ for which $\rho(x_n, y_n) < 2^{-n}$. It remains to put $C_x = \{x\} \cup \bigcup\{O_n \colon x_n = x\}$. Now it is straightforward to verify that the system satisfies conditions mentioned in the claim.

Realize that for any pair $y, z \in X^\delta \setminus X^{\delta+1}$ we have rank $C_y = \delta + 1 =$ rank C_z and top $C_y = \{y\} \cong \{z\} =$ top C_z. Thus according to Lemma 3.7 there exists a homeomorphism $\varphi_{y,z} \colon C_y \to C_z$. Clearly $\varphi_{y,z}(y) = z$. Define a mapping $g \colon X \to X$ by the rule

$$g(x) = \begin{cases} h(x), & \text{if } x \in X^\delta, \\ \varphi_{y,h(y)}(x), & \text{if } x \in C_y. \end{cases}$$

The mapping g is a correctly defined bijection and it is continuous at the points from $\bigcup C_y$. Now take an arbitrary point $x \in X^{\delta+1}$ and let us verify continuity of g at this point. Clearly $g\restriction_{X^\delta} = h$ is continuous at the point x. Thus it is enough to verify continuity of g with respect to the set $X \setminus X^\delta$. Let $x_n \in X \setminus X^\delta$ be a sequence which tends to x. For every $n \in \omega$ we are able to find $y_n \in X^\delta \setminus X^{\delta+1}$ such that $x_n \in C_{y_n}$. Since diam C_{y_n} tends to zero we get that the sequence y_n tends to x. Further since h is continuous we derive that $h(y_n) = g(y_n)$ tends to $h(x) = g(x)$. Moreover we know that $g(x_n) \in C_{h(y_n)}$ and thus $g(x_n)$ converges to $g(x)$ since the diameters of $C_{h(y_n)}$ tend to zero.

Hence we have just proved that g is a continuous one-to-one mapping of X onto X extending the mapping h. Since the space X is compact we deduce that g is a homeomorphism.

It remains to deal with the case of a non-compact space X. We consider its one-point compactification $\alpha X = X \cup \{\infty\}$ and distinguish two cases.

- $(\alpha X)^\delta = X^\delta$. Then we can extend given homeomorphism $h \in \mathcal{H}(X^\delta)$ into a homeomorphism $g \in \mathcal{H}(\alpha X)$ because of the previous step. By Lemma 3.8 there is a homeomorphism $g' \in \mathcal{H}(\alpha X)$ such that $g'\big(g(\infty)\big) = \infty$ and which equals to identity on $(\alpha X)^\delta$. Consequently $g'g\restriction_X \in \mathcal{H}(X)$ extends h.

- $(\alpha X)^\delta = X^\delta \cup \{\infty\}$. Thus we can extend homeomorphism $h \in \mathcal{H}(X^\delta)$ into a homeomorphism $h' \in \mathcal{H}\big((\alpha X)^\delta\big)$ simply by $h'(\infty) = \infty$ and use the preceding step to obtain a homeomorphism $g \in \mathcal{H}(\alpha X)$ extending h'. Finally $g\restriction_X \in \mathcal{H}(X)$ is the desired homeomorphism extending h.

□

Theorem 3.11. *Let X be a countable locally compact space. Then the only H-compactifications of X are of the form $\beta X/F_\delta$ where*

$$F_\delta = \bigcap\big\{\overline{X^\epsilon} \colon \epsilon < \delta\big\} \cap X^*$$

for $1 \leq \delta \leq rank_c X + 1$.

Thus the lattice of all H-compactifications of X forms a chain isomorphic to an ordinal $rank_c X + 1$ when $rank_c X$ is finite and to an ordinal $rank_c X + 2$ when $rank_c X$ is infinite.

Proof. Realize that the Čech-Stone compactification of X is obtained as $\beta X/F^\delta$ for $\delta = \mathrm{rank}_c X + 1$. Suppose γX is an H-compactification distinct from βX and denote by $f \colon \beta X \to \gamma X$ the only extension of identity on X. Let δ be the least non-zero ordinal number for which there exists two distinct points $x \in \beta X \setminus \overline{X^\delta}$ and $y \in \beta X$ such that $f(x) = f(y)$. Note that both x and y are elements of the remainder and clearly $\delta \leq \mathrm{rank}_c X$ since $\overline{X^{\mathrm{rank}_c X}} \subseteq X$. Our aim is to prove that γX is equivalent to $\beta X/F_\delta$.

Lets take any point $z \in F_\delta$ distinct from x and y. We would like to show that arbitrary neighbourhood U of z contains a point which is mapped to $f(y)$ by f. Thus take a clopen neighbourhood O of the point x such that $O \cap X^\delta = \emptyset$ and $y, z \notin O$ and a clopen neighbourhood P of the point z which is a subset of U and does not contain y neither intersects O. Distinguish two possibilities in order to define a sequence $Z = \{z_n \colon n \in \omega\}$.

- If $\delta = \epsilon + 1$ define $\{z_n \colon n \in \omega\}$ to be a sequence contained in $P \cap (X^\epsilon \setminus X^\delta)$ whose limit is $\infty \in \alpha X$. This is possible since $P \cap X^\epsilon$ is closed and noncompact subset of X and $X^\epsilon \setminus X^\delta$ is dense in X^ϵ.
- When δ is a supremum of ordinals $\{\delta_n \colon n \in \omega\}$ less than δ it is possible to find a sequence $\{z_n \colon n \in \omega\}$ whose limit is $\infty \in \alpha X$ and such that $z_n \in P \cap X^{\delta_n} \setminus X^{\delta_n + 1}$.

At this moment let us mention that

$$\emptyset \neq \overline{Z} \setminus Z \subseteq \bigcap_{\epsilon < \delta} \overline{X^\epsilon} \setminus X^\delta. \tag{3.12}$$

Since Z and X^δ are two disjoint closed subsets of X there exists a clopen set $N \subseteq X$ containing the sequence Z and being disjoint from X^δ. Let us denote by M the clopen set $O \cap X$. It follows that rank $M = \mathrm{rank}_c M = \delta = \mathrm{rank}_c N =$ rank N. Moreover if δ is an isolated ordinal number we get that top $M \cong \omega \cong$ top N otherwise top $M = \emptyset =$ top N. Thus by Proposition 3.9 we get that the sets M and N are homeomorphic. Since they are clopen and disjoint we can define a homeomorphism $h \in \mathcal{H}(X)$ such that $h(M) = N$,

$h(N) = M$ and h is equal to identity on the complement of $M \cup N$. Since $x \in \overline{M}$ we get that $\beta h(x) \in U$ and $\beta h(y) = y$. There exists an extension $g \in \mathcal{H}(\gamma X)$ of the homeomorphism h, because γX is an H-compactification. It holds that $gf = f\beta h$ and since $f(x) = f(y)$ we get that $f\big(\beta h(x)\big) = f\big(\beta h(y)\big) = f(y)$. Consequently $\beta h(x)$ is a point contained in U which is mapped by f to $f(y)$. Remember that the U was arbitrary neighbourhood of z thus by continuity of f we get $f(z) = f(y)$.

We have just proved that $\gamma X \leq \beta X / F_\delta$. If we assume this inequality is strict we get a contradiction with the minimality of δ.

In order to prove that all mentioned H-compactifications are mutually distinct we have to verify that $F_\epsilon \neq F_\delta$ for $1 \leq \epsilon < \delta \leq \mathrm{rank}_c X + 1$. To do this is remains to verify that $\overline{X^\delta} \setminus X$ is a proper subset of $\bigcap_{\epsilon<\delta} \overline{X^\epsilon} \setminus X$ for $1 \leq \delta \leq \mathrm{rank}_c X$ which is a consequence of 3.12. □

Example 3.13. The space $\omega \times (\omega + 1)^n$ admits exactly $n + 2$ H-compactifications.

Remark 3.14. When dealing with countable non-locally compact spaces the situation seems to be more complicated even in the range of metrizable spaces.

3.3 Euclidean spaces

Van Douwen proved in [vD79] that there exist only three H-compactifications of the real line. Now, we will continue in this research. We show that there are only two H-compactifications of Euclidean spaces of heigher dimension.

Theorem 3.15 (van Douwen [vD79])**.** *There are exactly two H-compactifications of the space* $[0, +\infty)$*.*

Corollary 3.16 (van Douwen [vD79])**.** *The only H-compactifications of the real line* $\mathbb{R}$ *are* $\alpha\mathbb{R}$*,* $[-\infty, +\infty]$ *and* $\beta\mathbb{R}$*.*

The following definitions are given in order to express Lemma 3.19 in a shorter way. The lemma and Theorem 3.21 are formulated for a general space X which satisfies certain conditions instead of the Euclidean space $\mathbb{R}^n$. This is mainly because of the fact that the proof seems to be more transparent then.

Definition 3.17. If $\mathcal{U}$ is a collection of subsets of a metric space X then the *mesh* of $\mathcal{U}$ is the number

$$\mathrm{mesh}\, \mathcal{U} = \sup\{\mathrm{diam}\, U : U \in \mathcal{U}\} \in [0, +\infty].$$

Definition 3.18. We say that an open set U of a space X satisfies *property* $(*)$ if whenever we take a set $E \subseteq U$ closed in X and $V \subseteq U$ open and non-empty, there is a homeomorphism $h \in \mathcal{H}(X)$ such that $h(E) \subseteq V$ and h is equal to identity on the complement of U.

Lemma 3.19. *Let X be a separable locally compact metric space and suppose that there exists $N \in \omega$ such that for every $\epsilon > 0$ there is an open cover $\mathcal{U}$ which can be expressed as a union of N discrete subsystems with mesh less than ϵ, where every set $U \in \mathcal{U}$ has property $(*)$. Let $M = 2N$. Then*

(i) for every closed set F contained in an open set H there exist a closed discrete set $C \subseteq F$ and closed sets $F_0, \ldots, F_{M-1}$ such that $F = \bigcup F_i$ and for every $i < M$ and open neighbourhood G of C there is a homeomorphism $h \in \mathcal{H}(X)$ such that $h(F_i) \subseteq G$ and h is equal to identity on the complement of H.

(ii) for every pair of disjoint closed sets $F, F' \subseteq X$ there exist disjoint closed discrete sets $C, C' \subseteq X$ and closed sets $F_0, \ldots, F_{M-1}$, $F'_0, \ldots, F'_{M-1}$ such that $F = \bigcup F_i$, $F' = \bigcup F'_j$ and for arbitrary $i, j < M$ and neighbourhoods G and G' of C and C' respectively there exists a homeomorphism $h \in \mathcal{H}(X)$ such that $h(F_i) \subseteq G$ and $h(F'_j) \subseteq G'$.

Proof. (i) Fix a closed set F and an open set H containing F. Our first claim is that there exists an open cover $\mathcal{V}$ of F in H which can be expressed as a union of M discrete systems which consist of sets with property $(*)$.

Let us find a chain of compact sets $\{K_n \colon n \in \omega\}$ where every set K_n is contained in the interior of K_{n+1} and $K_0 = \emptyset$. This can be done since the space X is locally compact and of countable weight. Denote by d_n the distance of K_n from $K_{n+2} \setminus \operatorname{int} K_{n+1}$ and by ϵ_n arbitrary positive number less than $\frac{1}{3}\min\{d_0, \ldots, d_n\}$ and $\operatorname{dist}(F \cap K_n, X \setminus H)$. There exists an open cover $\mathcal{U}^n = \mathcal{U}^n_0 \cup \cdots \cup \mathcal{U}^n_{M-1}$ of X of mesh less than ϵ_n, consisting of sets with property $(*)$ where every system $\mathcal{U}^n_i$ is discrete. Define now

$$\mathcal{V}_i = \bigcup_{n \text{ is odd}} \{U \in \mathcal{U}^{n+1}_i \colon U \cap F \cap (K_{n+1} \setminus \operatorname{int} K_n) \neq \emptyset\}$$

for $i < N$ and

$$\mathcal{V}_i = \bigcup_{n \text{ is even}} \{U \in \mathcal{U}^{n+1}_{i-N} \colon U \cap F \cap (K_{n+1} \setminus \operatorname{int} K_n) \neq \emptyset\}$$

for $N \leq i < M$.

The systems $\mathcal{V}_i$ are discrete and consist of sets with property $(*)$. A collection $\mathcal{V} = \bigcup_{i<M} \mathcal{V}_i$ is the required cover of F contained in H. Thus the claim is proved.

Since metrizable spaces are paracompact according to Stone theorem [Eng89, p. 300] we can find closed indexed refinement $\{E(V) \colon V \in \mathcal{V}\}$ whose union is F, because by Remark 5.1.7 from [Eng89, p. 301] every open cover of a regular paracompact space has a closed indexed refinement. Observe that a set $F_i = \bigcup\{E(V) \colon V \in \mathcal{V}_i\}$ is closed, because it is a union of a discrete system of closed sets. Suppose that C is a selecting set from the system $\{E(V) \colon V \in \mathcal{V}\}$. This is a closed discrete set.

Suppose that G is an open neighbourhood of C and fix $i < M$. For every $V \in \mathcal{V}_i$ there exists a homeomorphism $h_V \in \mathcal{H}(X)$ which sends the set $E(V)$ into $G \cap V$ and which is equal to identity on the complement of the set V. This is because V has the property $(*)$.

The fact that the collection $\mathcal{V}_i$ is discrete allows us to construct a homeomorphism $h \in \mathcal{H}(X)$ which is roughly speaking a composition of all these h_U. Consequently h sends the set F_i into G.

(ii) For given disjoint closed sets F and F' we are able to find open sets H and H' whose closures are disjoint and which contain F and F' respectively. By part (i) which is used twice there exists closed discrete sets C and C' and closed sets $F_0, \dots, F_{M-1}$, $F'_0, \dots, F'_{M-1}$ such that whenever we take $i, j < M$ and neighbourhoods G and G' of C and C' respectively there exist homeomorphisms $h, h' \in \mathcal{H}(X)$ such that $h(F_i) \subseteq G$, $h'(F'_j) \subseteq G'$, $h\restriction_{X \setminus H} = \mathrm{id}$ and $h'\restriction_{X \setminus H'} = \mathrm{id}$. It suffices to put $h'' = h'h$ which is the required homeomorphism. □

Definition 3.20. Let X be a topological space and suppose that $\mathcal{P}$ is a collection of some subspaces of X. A topological space X is called *n-homogeneous with respect to $\mathcal{P}$* if for any sets $C_i, D_i \in \mathcal{P}$ for $i < n$ satisfying $C_i \cong D_i$ and $C_i \cap C_j = \emptyset = D_i \cap D_j$ for $i \neq j$ there exists a homeomorphism $h \in \mathcal{H}(X)$ such that $h(C_i) = D_i$ for every $i < n$.

When $n = 1$ we omit the number and we write just *homogeneous with respect to $\mathcal{P}$*. Note that for example homogeneity with respect to points is equivalent to the classical notion of homogeneity.

Theorem 3.21. *Let X be a non-compact space satisfying assumptions of Lemma 3.19 which is 2-homogeneous with respect to closed discrete sets. Then αX and βX are the only H-compactifications of X.*

Proof. Suppose that γX is an H-compactification of X distinct from the one-point compactification. Let us take arbitrary disjoint closed sets F and F' in X. Our aim is to prove that their closures in γX are disjoint too, because then we get that γX is equivalent to βX.

By Lemma 3.19 there exist closed discrete sets C, C' and closed sets $F_0, \dots, F_M, F'_0, \dots, F'_M$ with properties mentioned above. Since $F = \bigcup F_i$ and $F' = \bigcup F'_j$ it is enough to prove that for arbitrary $i, j < M$ the closures of F_i and F'_j are disjoint in γX.

Realize that there exist two countable infinite closed discrete sets D and D' of X whose closures in γX are disjoint because the compactification γX has remainder containing at least two points. Since the space X is 2-homogeneous with respect to closed discrete sets and since γX is an H-compactification we get that closures of C and C' in γX are disjoint too. Hence we can separate them by open sets G and G' in X whose closures in γX are disjoint.

Because of Lemma 3.19 we can find a homeomorphism $h \in \mathcal{H}(X)$ for which $h(F_i) \subseteq G$ and $h(F'_j) \subseteq G'$. Consequently the closures of $h(F_i)$ and $h(F'_j)$ in γX are disjoint and since γX is an H-compactification we get that the closures of F_i and F'_j are disjoint too.

Thus we have just proved that γX is equivalent to βX. □

What we need to show is that Euclidean spaces of dimension at least two satisfy assumptions of Theorem 3.21. This is partially done in Proposition 3.23. The following notion will be useful when dealing with proof.

Definition 3.22. A space X is called *strongly locally homogeneous* if for every point $x \in X$ and arbitrary neighbourhood U of x there exists a neighbourhood V of x in U such that for every $y \in V$ we can find a homeomorphism of X which sends the point x to y and is equal to identity on the complement of V.

Note that if we have an open connected set U in a strongly locally homogeneous space X and two points $x, y \in U$, there is always a homeomorphism $h \in \mathcal{H}(X)$ for which $h(x) = y$ and which equals to identity on the complement of U. This is because the set $\{g(x) \colon g \in \mathcal{H}(X), g\restriction_{X\setminus U} = \mathrm{id}\}$ is clopen in U and thus it has to be equal to the set U.

It is a well-known fact that Euclidean spaces are strongly locally homogeneous (see [vM01, p. 64]).

Proposition 3.23. *Let $n \geq 2$. Then every bijection of closed discrete subsets of the space $\mathbb{R}^n$ can be extended to a homeomorphism of the whole space.*

Proof. Let C and D be the given closed discrete subsets of cardinality $m \leq \omega$ of the space $\mathbb{R}^n$ and $b \colon C \to D$ a bijection. Next we will use the structure of a normed linear space $\big(\mathbb{R}^n, \|\cdot\|\big)$. The words *span* and *conv* mean *linear* and *convex hull* respectively. Let S be a set consisting of those points from $\mathbb{R}^n$ either which lies on a line determined by two points from $C \cup D$ or whose distance from two distinct points from $C \cup D$ is the same. This set is a countable union of nowhere dense sets thus S is a set of the first category in $\mathbb{R}^n$. Hence by the Baire theorem there is a point in the difference $\mathbb{R}^n \setminus S$ and we can without loss of generality assume that $0 \in \mathbb{R}^n \setminus S$.

Enumerate now the set C as $\{c_i \colon i < m\}$ and denote $r_i = \|c_i\|$, $\varphi_i = \frac{c_i}{\|c_i\|} \in \mathbb{S}^{n-1}$, $d_i = b(c_i)$ and $s_i = \|d_i\|$. Because of the first paragraph we get that $r_i \neq r_j$, $\varphi_i \neq \varphi_j$ and $s_i \neq s_j$ for $i < j < m$. Let us fix $i < m$ for a while. Denote by ϵ_i the positive distance of the set $\bigcup\{\mathrm{span}(0, c) \colon c \in C, \|c\| < r_i\} \cup C \setminus \{c_i\}$ from the segment $\mathrm{conv}(c_i, s_i\varphi_i)$. Denote by U_i the set of all points whose distance from $\mathrm{conv}(c_i, s_i\varphi_i)$ is less than $\frac{1}{3}\epsilon_i$. Since the space $\mathbb{R}^n$ is strongly locally homogeneous and the open set U_i is connected it follows that there exists a homeomorphism $h_i \in \mathcal{H}(\mathbb{R}^n)$ such that $h_i(c_i) = s_i\varphi_i$ and which equals to identity on the complement of U_i. Since the system $\{U_i \colon i < m\}$ is discrete we can define a homeomorphism $h \in \mathcal{H}(\mathbb{R}^n)$ for which

$$h(x) = \begin{cases} h_i(x), & \text{if } x \in U_i, \\ x, & \text{otherwise.} \end{cases}$$

We proceed by a similar method to find a homeomorphism h'. Let ϵ'_i be a positive distance of the point s_i from the set $\{s_j \colon j \neq i\}$ and denote by U'_i the set of all points whose distance from $s_i\mathbb{S}^{n-1}$ is less than $\frac{1}{3}\epsilon'_i$. We can find a homeomorphism $h'_i \in \mathcal{H}(\mathbb{R}^n)$ such that $h'_i(s_i\varphi_i) = d_i$ and which is equal to identity at the complement of the set U'_i. Since the system $\{U'_i \colon i < m\}$ is discrete it is correct to define a homeomorphism $h' \in \mathcal{H}(\mathbb{R}^n)$ by a formula

$$h'(x) = \begin{cases} h'_i(x), & \text{if } x \in U'_i, \\ x, & \text{otherwise.} \end{cases}$$

Now it is enough to define h'' as a composition $h'h$ to get the desired homeomorphism for which $h''(c_i) = h'h(c_i) = h'(s_i\varphi_i) = d_i = b(c_i)$. □

Remark 3.24. The proof of Proposition 3.23 can be done more easily for Euclidean spaces of dimension at least three. In the plane the situation is more complicated. Note that this proposition fails to be true for $n = 1$ obviously.

Corollary 3.25. *There are exactly two H-compactifications of $\mathbb{R}^n$ for $n \geq 2$. Namely $\alpha\mathbb{R}^n$ and $\beta\mathbb{R}^n$.*

Proof. It is enough to realize that the assumptions of Theorem 3.21 are satisfied. The fact that $\mathbb{R}^n$ is 2-homogeneous with respect to closed discrete sets follows easily from Proposition 3.23. For verification of the assumptions of Lemma 3.19 consider a maximum metric ρ of the space $\mathbb{R}^n$ and denote by $B_\rho(x, r)$ an open ball with center x and diameter r. Put $N = 2^n$. For arbitrary $\epsilon > 0$ we can take a cover $\mathcal{U} = \bigcup\{\mathcal{U}_j \colon j \in 2^n\}$ of $\mathbb{R}^n$ where

$$\mathcal{U}_j = \left\{ B_\rho\left(\frac{\epsilon}{3}i, \frac{\epsilon}{4}\right) : i \in \mathbb{Z}^n,\ i_k \equiv j_k \pmod 2 \right\}.$$

All collections $\mathcal{U}_j$ are discrete and they consist of sets with property $(*)$, because every ball $B_\rho(x, r)$ in $\mathbb{R}^n$ has property $(*)$.

For completeness we should add that the two H-compactifications are distinct which is however an immediate consequence of the fact that there exists a continuous bounded function which hasn't limit at infinity. □

Suppose now that γX is an H-compactification of a space X. The property of X being homogeneous with respect to closed discrete sets implies that there is at most one point in $\gamma X \setminus X$ with countable character in γX. Especially, we get the following result.

Proposition 3.26. *Let X be a non-compact Tychonoff space which is homogeneous with respect to closed discrete sets. Then X admits at most one first countable H-compactification and that is one-point compactification.*

Proof. When two points x and y from the remainder of a first countable H-compactification γX are given, we are able to find sequences x_n and y_n in X whose limits are x and y respectively. Thus the sets $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ are infinite closed in X and discrete. Since the space X is homogeneous with respect to sets of this type there exists a homeomorphism $h \in \mathcal{H}(X)$ sending a set $\{x_n : n \in \omega\}$ onto a set $\{x_n, y_n : n \in \omega\}$. Suppose now that $g \in \mathcal{H}(\gamma X)$ is a continuous extension of h. From continuity we get that $x = g(x) = y$. Thus we have just proved that $\gamma X \setminus X$ contains at most one point. $\square$

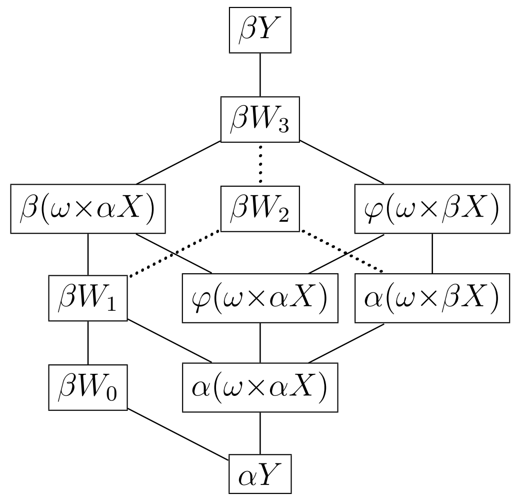
3.4 Products of omega and Euclidean space

In this section we are going to describe all H-compactifications of a product $\omega \times \mathbb{R}^n$. It is shown that there is always finitely many of them, but the situation is different in case $n = 1$ from the case $n \geq 2$. Some of these H-compactifications can be obtained via applying compactifications α, β and φ, where φ denotes so called *Freudenthal compactification* which is the largest compactification with zero-dimensional remainder. For a locally compact space X it always exists and can be described as a quotient of βX where all components of $\beta X \setminus X$ shrink into points. Some additional notes concerning Freudenthal compactification can be found in [Dom03].

We use Hasse diagrams in order to describe a lattice or a partially ordered set. Here, two objects are connected with a line if the upper one is a minimal element strictly bigger than the lower one. For more information about Hasse diagrams we refer to the first chapter of [Bir67, p. 4].

The following result can be applied to $X = \mathbb{R}^n$ where $n \geq 2$ since the assumptions were verified in the proof of Corollary 3.25.

Theorem 3.27. *Let X be a non-compact connected space satisfying the same assumptions as in Lemma 3.19 which is 2-homogeneous with respect to closed discrete sets. Then the space $Y = \omega \times X$ admits exactly eleven H-compactifications which form a lattice described below.*



Here W_0 is a quotient space $\omega \times \alpha X/\big(\omega \times \{\infty\}\big)$. A space W_1 is obtained via addition of one point ∞ to a space $\omega \times \alpha X$ where base neighbouhoods of ∞ are of the form

$$\{\infty\} \cup \bigcup_{n>m} \{n\} \times \big(\alpha X \setminus K_n\big)$$

where K_n are compact subsets of X and $m \in \omega$. A space W_2 arise by adjunction of a point ∞ to the space Y. Base neighbourhoods of ∞ are of the form

$$\{\infty\} \cup \bigcup_{n>m} \{n\} \times \big(X \setminus K_n\big)$$

where K_n are compact subsets of X and $m \in \omega$. Finally a space W_3 is given by $Y \cup \omega^*$ where Y is an open set in W_3 and base neighbourhoods of a point $x \in \omega^*$ are of the form

$$\big(\omega^* \cap \overline{M}^{\beta\omega}\big) \cup \bigcup_{n\in\omega} \{n\} \times \big(X \setminus K_n\big)$$

where $M \subseteq \omega$, $x \in \overline{M}^{\beta\omega}$ and K_n are compact subsets of X.

Notation 3.28. We will use a consequence of Theorem 2.3 that all H-compactifications of the (locally compact) space Y are in a natural one-to-one correspondence with those closed equivalences $E \subseteq (\beta Y \setminus Y)^2$ such that for every homeomorphism $h \in \mathcal{H}(Y)$ we have $\beta h \times \beta h(E) = E$. Equivalences with this property are called *invariant* equivalences. Note that closed invariant equivalences of a locally compact space form a lattice antiisomorphic to the lattice of all H-compactifications.

Denote by $\pi\colon Y \to \omega$ the projection onto the first coordinate, by X_n a set $\{n\} \times X$, by C a closed set

$$\beta Y \setminus \bigcup \left\{ \overline{K}^{\beta Y} : K \subseteq Y, \text{ for every } n \text{ the set } K \cap X_n \text{ is compact} \right\}$$

and by Δ a diagonal of the square $(\beta Y \setminus Y)^2$.

Moreover, define three equivalences on Y^*, indexed by symbols from a set $\{C, *, =\}$, as follows.

$$\begin{aligned}
E_C &= \Delta \cup \{(x, y) \in C \times C\} \\
E_* &= \Delta \cup \{(x, y) \in Y^* \times Y^* \colon \beta\pi(x),\ \beta\pi(y) \in \omega^*\} \\
E_= &= \Delta \cup \{(x, y) \in Y^* \times Y^* \colon \beta\pi(x) = \beta\pi(y)\}
\end{aligned}$$

It can be verified that these equivalences are closed. Since the space X is connected we get that the mapping π is homogeneous. Consequently all the three equivalences above are invariant.

Proof. It is interesting to mention that in fact those four equivalences $E_C, E_*, E_=$ and Δ generate the lattice of all closed invariant equivalences on Y^*, as will be shown. We denote for any $S \subseteq \{C, *, =\}$ by E_S a closed invariant equivalence $\bigwedge_{i \in S} E_i$. Our claim is that the only closed one-generated invariant equivalences are these E_S except $E_=$. Thus take a pair $(x, y) \in (\beta Y \setminus Y)^2$ and realize that if $x = y$ then this pair generates Δ, otherwise we have to distinguish following possibilities. We give a precise proof only in case of equivalence $E_{C*=}$ since other cases are very similar.

- $x, y \in C$
  - $\beta\pi(x), \beta\pi(y) \in \omega^*$
    - $\beta\pi(x) = \beta\pi(y)$ In order to prove that the pair (x, y) generates the equivalence $E_{C*=}$ take arbitrary couple $(x', y') \in E_{C*=} \setminus \Delta$ and arbitrary neighbourhood $G \times H$ of the point (x', y') in $\beta Y \times \beta Y$. We want to find a homeomorphism $h \in \mathcal{H}(Y)$ for which $\big(\beta h(x), \beta h(y)\big) \in \overline{G} \times \overline{H}$. Since $x', y' \in C$ and $\beta\pi(x') = \beta\pi(y') \in \omega^*$ there exists an infinite set $K \in \beta\pi(x')$ such that for every $n \in K$ the closures of the sets $G \cap X_n$ and $H \cap X_n$ in Y are non-compact. We can assume that $\omega \setminus K$ is infinite. Suppose that E and F are disjoint closed subsets of $L \times X$ for which $x \in \overline{E}$ and $y \in \overline{F}$ where L is an infinite subset of ω such that $\omega \setminus L$ is infinite. Let $b\colon \omega \to \omega$ be an arbitrary bijection for which $b(L) = K$.

For fixed $n \in L$ apply Lemma 3.19 to the space X_n and closed sets $E \cap X_n$ and $F \cap X_n$ in order to get closed discrete sets $C^n, D^n \subseteq X_n$ and closed sets $E_i^n, F_j^n \subseteq X_n$ where $i < M$ with certain properties. Since

$$E = \bigcup_{i<M} \bigcup_{n\in\omega} E_i^n \quad \text{and} \quad F = \bigcup_{j<M} \bigcup_{n\in\omega} F_j^n,$$

there exist indeces $i, j < M$ such that $x \in \overline{\bigcup\{E_i^n : n \in \omega\}}$ and $y \in \overline{\bigcup\{F_j^n : n \in \omega\}}$. Since the space X is 2-homogeneous with respect to closed discrete sets we can find a homeomorphism $h_n : X_n \to X_{b(n)}$ such that $h_n(C^n) \subseteq G \cap X_{b(n)}$ and $h_n(D^n) \subseteq H \cap X_{b(n)}$. When we use the result of Lemma 3.19 we get that there are homeomorphisms $g_n \in \mathcal{H}(X_n)$ such that $g_n(E_i^n) \subseteq h_n^{-1}\big(G \cap X_{b(n)}\big)$ and $g_n(F_j^n) \subseteq h_n^{-1}\big(H \cap X_{b(n)}\big)$.

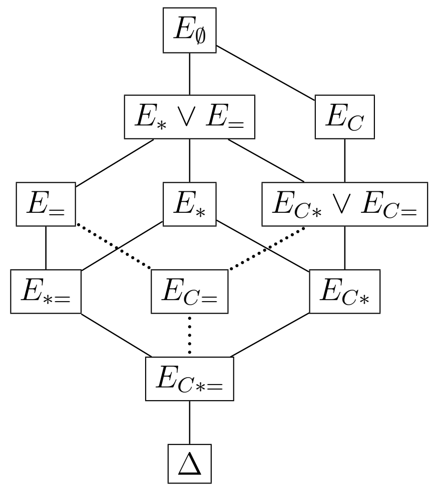
For $n \in \omega \setminus L$ we put h_n to be the natural homeomorphism $X_n \to X_{b(n)}$ and g_n to be the identity on X_n. Now $h = \bigcup\{h_n g_n : n \in \omega\} \in \mathcal{H}(Y)$ is the desired homeomorphism.

  - $\beta\pi(x) \neq \beta\pi(y)$. In this case the pair (x, y) generates E_{C*}.
- $\beta\pi(x) \in \omega$ or $\beta\pi(y) \in \omega$
  - $\beta\pi(x) = \beta\pi(y)$. Consequently (x, y) generates $E_{C=}$.
  - $\beta\pi(x) \neq \beta\pi(y)$. The pair (x, y) generates E_C.

- $x \notin C$ or $y \notin C$
  - $\beta\pi(x), \beta\pi(y) \in \omega^*$
    - $\beta\pi(x) = \beta\pi(y)$. This implies that the pair (x, y) generates $E_{*=}$.
    - $\beta\pi(x) \neq \beta\pi(y)$. Then it can be shown that (x, y) generates E_*.
  - $\beta\pi(x) \in \omega$ or $\beta\pi(y) \in \omega$
    - $\beta\pi(x) = \beta\pi(y)$. This can not happen.
    - $\beta\pi(x) \neq \beta\pi(y)$. Then (x, y) generates $E_\emptyset$.

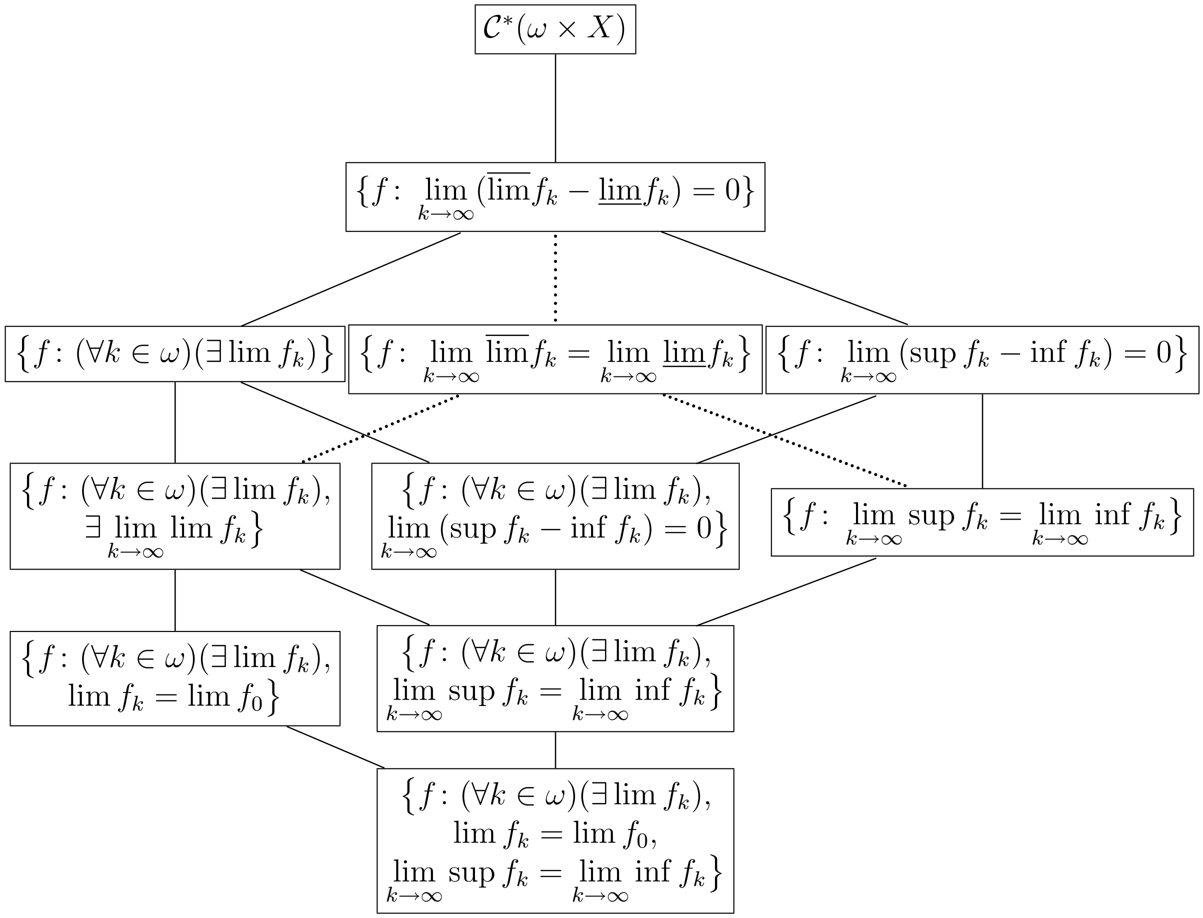
Using the previous claim we conclude that arbitrary closed invariant congruence can be obtained as a join of some subset of equivalences E_S and Δ. Thus there exist at most $2^9 = 512$ of them. However there are obvious inclusions $\Delta \subseteq E_S \subseteq E_T$ for $T \subseteq S \subseteq \{C, *, =\}$ which reduce this number rapidly.

Observe that $E_* = E_{*=} \vee E_{C*}$, because if we take $(x, y) \in E_* \setminus \Delta$ we can find $x', y' \in C$ such that $\beta\pi(x) = \beta\pi(x')$ and $\beta\pi(y) = \beta\pi(y')$ and thus $(x, x') \in E_{*=}$, $(x', y') \in E_{C*}$ and $(y', y) \in E_{*=}$, hence $(x, y) \in E_{*=} \vee E_{C*}$. Similarly it can be shown that $E_= = E_{*=} \vee E_{C=}$ and $E_\emptyset = E_C \vee E_{*=}$. If we put together these equalities and the obvious inclusions we get that the lattice of all closed invariant equivalences is given by the following Hasse diagram.



It remains to verify that this lattice corresponds to the lattice mentioned in the statement, but this is just a routine work. $\square$

Remark 3.29. Below we give a characterization of all H-compactifications of the space $\omega \times X$ from Theorem 3.27 using corresponding rings of continuous functions. For simplicity we denote by f_k the restriction $f{\restriction}_{\{k\}\times X}$ whenever f is an element of $\mathcal{C}^*(\omega \times X)$. The symbol $\lim f_k$ means $\lim\limits_{x\to\infty} f_k(x)$ where $X \cup \{\infty\}$ is the one-point compactification of the space X. The symbols $\overline{\lim}$ and $\underline{\lim}$ denote limes superior and limes inferior respectively.



Corollary 3.30. *The only H-compactifications of the space $Z = \omega \times \mathbb{S}^n$ for $n \geq 2$ are Alexandroff one-point compactification αZ, Freudenthal compactification φZ and Čech-Stone compactification βZ.*

Proof. Since there is a homogeneous embedding of the space $\omega \times \mathbb{R}^n$ onto a dense subspace of the product $\omega \times \mathbb{S}^n$, we get that every H-compactification of the space $\omega \times \mathbb{S}^n$ is at the same time an H-compactification of the space $\omega \times \mathbb{R}^n$. From Theorem 3.27 we derive that this is possible only in three described cases. $\square$

This situation is different in case $n = 1$. Van Douwen noted in [vD79] that there exist at least eleven H-compactifications of the product $\omega \times \mathbb{R}$. In fact there is exactly 26 of them. However it would be a long-distance run to prove it completely, because there is a lot of routine work. Thus we omit some details in the proof.

Theorem 3.31. *There exist exactly 26 H-compactifications of the space $\omega \times \mathbb{R}$. They form a lattice described in diagram 3.33.*

Proof. Let us use notation described in 3.28 where the space X is substituted by $\mathbb{R}$. Moreover, for any pair E, F of closed subsets of $\mathbb{R}$ define $o(E, F)$ to be infimum of all $k \in \omega$ such that there exist intervals (i.e. connected sets) $I_0, \ldots, I_k$ covering E and not intersecting F. Note that $o(E, F) = +\infty$ whenever E and F have non-empty intersection. By a simple consideration we derive that numbers $o(E, F)$ and $o(F, E)$ are either both infinite or they are both finite and their difference is at most one. For any pair $x, y \in \beta Y$ denote

$$\text{over}(x, y) = \inf\Big\{\sup_{n\in\omega} o(E \cap X_n, F \cap X_n) \colon x \in \overline{E}, y \in \overline{F}, E, F \subseteq Y\Big\}$$

and define the following one closed invariant equivalence $E_\infty = \{(x, y) \in Y^* \times Y^* \colon \text{over}(x, y) = \infty\}$. For any set $S \subseteq \{C, *, =, \infty\}$ put $E_S = \bigwedge_{i\in S} E_i$. Thus we get in this way at most sixteen equivalences but it is easily seen that $E_\infty \subseteq E_=$. Hence there are only twelve of them.

Our claim is that all one-generated closed invariant equivalences are of the form E_S for some set $S \subseteq \{C, *, =, \infty\}$ and Δ. To verify this take arbitrary pair $(x, y) \in Y^* \times Y^*$ and denote by E a closed invariant equivalence generated by this couple. If $x = y$ we get $E = \Delta$. Otherwise if $x \neq y$ we distinguish these possibilities. An argumentation at each step is necessary, but it is omited.

- $x, y \in C$
  - $\beta\pi(x), \beta\pi(y) \in \omega^*$
    - $\beta\pi(x) = \beta\pi(y)$
      - over$(x, y) = \infty$: $E = E_{C*=\infty}$
      - over$(x, y) < \infty$: $E = E_{C*=}$
    - $\beta\pi(x) \neq \beta\pi(y)$: $E = E_{C*}$
  - $\beta\pi(x) \in \omega$ or $\beta\pi(y) \in \omega$
    - $\beta\pi(x) = \beta\pi(y)$
      - over$(x, y) = \infty$: $E = E_{C=\infty}$
      - over$(x, y) < \infty$: $E = E_{C=}$
    - $\beta\pi(x) \neq \beta\pi(y)$: $E = E_C$
- $x \notin C$ or $y \notin C$
  - $\beta\pi(x), \beta\pi(y) \in \omega^*$
    - $\beta\pi(x) = \beta\pi(y)$

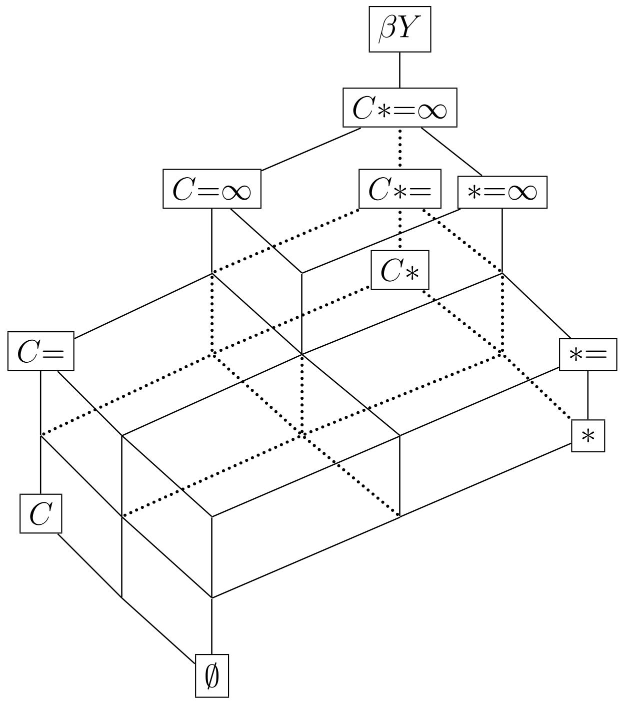
    - over$(x,y)=\infty$: $E=E_{*=\infty}$
    - over$(x,y)<\infty$: $E=E_{*=}$
  - $\beta\pi(x)\neq\beta\pi(y)$: $E=E_{*}$
- $\beta\pi(x)\in\omega$ or $\beta\pi(y)\in\omega$
  - $\beta\pi(x)=\beta\pi(y)$: This cannot happen.
  - $\beta\pi(x)\neq\beta\pi(y)$: $E=E_{\emptyset}$

Next, a partially ordered set of all one-generated closed invariant equivalences is given (this is not a lattice).

$$
\begin{tikzcd}
 & & E_{\emptyset} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & E_{*} \arrow[d, dash] \arrow[dr, dash] & & E_{C} \arrow[dl, dash] \arrow[d, dash] & \\
 & E_{*=} \arrow[d, dash] \arrow[dr, dash] & E_{C*} \arrow[d, dash] & E_{C=} \arrow[dl, dash] \arrow[d, dash] & \\
 & E_{*=\infty} \arrow[dr, dash] & E_{C*=} \arrow[d, dash] & E_{C=\infty} \arrow[dl, dash] & \\
 & & E_{C*=\infty} \arrow[d, dash] & & \\
 & & \Delta & &
\end{tikzcd}
\tag{3.32}
$$

Remember that every closed invariant equivalence is a join of some collection of those 11 equivalences from diagram 3.32. It can be counted by hand that there is 15 (unordered) pairs of equivalences in 3.32 that are not comparable, only 5 incomparable triples and clearly no such quadruples. By virtue of three equalities $E_{*}=E_{C*}\vee E_{*=}$ and $E_{\emptyset}=E_{C}\vee E_{*}=E_{C}\vee E_{*=}$ we get that there is in fact at most $12=15-3$ two-generated invariant closed congruences that are not one-generated and at most 3 three-generated that are not two-generated. Thus there is at most $11+12+3=26$ H-compactifications.

Finally, we get a Hasse diagram of the lattice of all H-compactifications of the space $\omega\times\mathbb{R}$. For shortness we write for example '$C=$' instead of $\beta Y/E_{C=}$.



(3.33)

It remains just to verify that all these compactifications are pairwise non-equivalent. This is however a lot of routine work only. $\square$

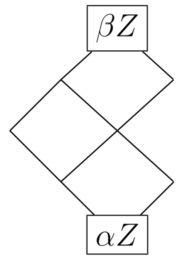
Remark 3.34. When dealing with the problem of finding all H-compactifications of $\omega \times \mathbb{R}$, one starts with compactifications obtained in a natural way in the form $\gamma(\omega \times \delta\mathbb{R})$ where γ and δ are substituted by α, β or φ. Even if we add $\alpha(\omega \times \mathbb{R})$ we obtain only 10 H-compactifications in this way.

Corollary 3.35. *There exist exactly four H-compactifications of the space $Z = \omega \times \mathbb{S}$. These are (from the least to the biggest) αZ, Freudenthal compactification φZ, a compactification γZ described below and βZ.*

The compactification γZ can be described as $\beta Z/_\sim$ where $x \sim y$ iff $\sup\{o(E_n, F_n) \colon n \in \omega\} = +\infty$ for every pair of closed sets $E, F \subseteq Y$ such that $x \in \overline{E}$ and $y \in \overline{F}$. In this situation $A_n = A \cap \{n\} \times \mathbb{S}$, for $A \subseteq Z$ and $o(E_n, F_n)$ is infimum of all $k \in \omega$ for which there exist connected sets $I_0, \ldots, I_k \subseteq Z_n$ such that $E_n \subseteq I_0 \cup \cdots \cup I_k \subseteq Z_n \setminus F_n$.

Proof. There is a homogeneous embedding $\omega \times \mathbb{R} \to \omega \times \mathbb{S}$ onto a dense subset and thus every H-compactification of the space $\omega \times \mathbb{S}$ forms at the same time an H-compactification of $\omega \times \mathbb{R}$. These were clasified in Theorem 3.31 from which it follows that only four of them are available. $\square$

Corollary 3.36. *The lattice of all H-compactifications of the space* $Z = \omega \times [-\infty, +\infty]$ *contains eight elements and is given by a Hasse diagram as follows.*



Proof. This is a consequence of Theorem 3.31 since there is a homogeneous embedding of $\omega \times \mathbb{R}$ onto a dense subset of Z. Exactly eight compactifications from the lattice 3.33 agree. $\square$

Chapter 4

Characterization using algebras of continuous functions

Recall that the Čech-Stone compactification can be characterized among all compactifications of X in the language of correspondent rings of bounded continuous functions. Namely a compactification γX is Čech-Stone compactification iff the naturally defined mapping $\varphi\colon \mathcal{C}(\gamma X) \to \mathcal{C}^*(X)$ for which $\varphi(f) = f\gamma$ is surjective. Is it possible to characterize H-compactifications in terms of rings? We have the following proposition obtained simply by applying a contravariant functor $\mathcal{C}^*(-)$. Let us call a ring homomorphism $\varphi\colon R \to S$ *invariant* if for every automorphism $\chi\colon S \to S$ there exists an automorphism $\psi\colon R \to R$ such that $\varphi\psi = \chi\varphi$.

Proposition 4.1. *Let γX be a compactification of a Tychonoff space X and denote by $\varphi\colon \mathcal{C}(\gamma X) \to \mathcal{C}^*(X)$ a ring homomorphism for which $\varphi(f) = f\gamma$. If φ is invariant then γX is an H-compactification.*

Proof. Suppose we are given $f \in \mathcal{C}^*(\gamma X)$ and $h \in \mathcal{H}(X)$. Define χ by the rule $\chi(g) = gh$ which is clearly an automorphism of the ring $\mathcal{C}^*(X)$. Because of the assumption the function $\chi\varphi(f) = fh$ is an element of image of φ and thus can be extended over γX. Hence by 2.3 we are done. $\square$

Example 4.2. Note that opposite implication in the previous proposition is no more true even for locally compact spaces. Choose X to be an open subset of $\beta\omega \setminus \{x, y\}$ containing ω properly where x and y are two distinct points from ω^*. If we apply contravariant functor $\mathcal{C}^*$ and realize that $\mathcal{C}^*(\omega)$ is isomorphic to $\mathcal{C}^*(X)$ we obtain the following diagram.

$$
\begin{tikzcd}
\omega \arrow[r, mapsto, "\mathcal{C}^*"] \arrow[d] & \mathcal{C}^*(\omega) \arrow[r, equal] & \mathcal{C}^*(X) & X \arrow[l, mapsto, "\mathcal{C}^*"'] \arrow[d] \\
\alpha X \arrow[r, mapsto, "\mathcal{C}^*"] & \mathcal{C}(\alpha X) \arrow[r, equal] \arrow[u, "\psi"'] & \mathcal{C}(\alpha X) \arrow[u, "\varphi"'] & \alpha X \arrow[l, mapsto, "\mathcal{C}^*"']
\end{tikzcd}
$$

Since αX is not an H-compactification of the space ω the mapping ψ is not invariant. The square in the middle of the diagram is commutative and that is why neither φ is invariant in spite of that αX is an H-compactification of X.

Chapter 5

Small H-compactifications

It is a natural question whether small spaces possess small compactifications. For example, it is well-known that a Tychonoff space of weight κ has a compactification of the same weight. Is it possible to find even an H-compactification of weight κ? As the space of rational numbers, which admits only one H-compactification [vD79], shows it is not possible in general. We have to add some other assumptions so that we could find an H-compactification of weight κ. To do this remind that compact spaces admit exactly one uniformity and that Čech uniformity of a Tychonoff space X is the uniformity inherited from βX.

Theorem 5.1. *Let X be a Tychonoff space of weight κ. Consider the space $\mathcal{H}(X)$ with uniform convergence topology with respect to Čech uniformity on X and suppose that its density doesn't exceed κ. Then there exists an H-compactification of the space X of weight κ.*

Proof. Without loss of generality suppose that the space X is a subset of a Tychonoff cube $T = [0,1]^\kappa$. Let $\mathcal{M}$ be a dense subset of $\mathcal{H}(X)$ of cardinality at most κ. We can assume moreover that $\mathcal{M}$ forms a group. Denote by $\gamma\colon X \to T^{\mathcal{M}}$ a mapping for which $\gamma(x) = \big(h(x)\big)_{h\in\mathcal{M}}$. Clearly γ is an embedding. Our aim is to show that $\overline{\gamma(X)}$ is the desired H-compactification.

First of all for any $h \in \mathcal{M}$ we define a mapping $\bar{h}\colon \overline{\gamma(X)} \to \overline{\gamma(X)}$ by the rule $\bar{h}\big((a_g)_{g\in\mathcal{M}}\big) = (a_{gh})_{g\in\mathcal{M}}$. The mapping $\bar{h}$ is well defined homeomorphism of $\overline{\gamma(X)}$. Moreover for any $x \in X$ we get

$$\bar{h}\gamma(x) = \bar{h}\big(g(x)_{g\in\mathcal{M}}\big) = \big(gh(x)\big)_{g\in\mathcal{M}} = \gamma h(x).$$

Thus we get $\bar{h}\gamma = \gamma h$.

Now we use density of $\mathcal{M}$ in $\mathcal{H}(X)$ to show that every $h \in \mathcal{H}(X)$ can be continuously extended over the compactification. By Theorem 3.2.1 in [Eng89, p. 136] it is enough to show that closures of $h^{-1}(E)$ and $h^{-1}(F)$ in γX are disjoint for disjoint closed sets E and F in γX. Denote by $\mathcal{U}$ the only uniformity on γX and note that $\mathcal{U}\restriction_X$ is coarser then the Čech uniformity on X. There exists a neighbourhood of diagonal $U \in \mathcal{U}$ for which $\overline{U[E]} \cap \overline{U[F]} = \emptyset$. From the density of $\mathcal{M}$ we get that there exists a homeomorphism $g \in \mathcal{M}$ for which $\big(h(x), g(x)\big) \in U$ for every $x \in X$. Then we obtain that

$$\overline{h^{-1}(E)} \cap \overline{h^{-1}(F)} \subseteq \overline{g^{-1}(U[E])} \cap \overline{g^{-1}(U[F])} \subseteq \bar{g}^{-1}(\overline{U[E]}) \cap \bar{g}^{-1}(\overline{U[F]}) = \emptyset.$$

□

A special case of Theorem 5.1 for metric spaces was proved in [dGM60]. The proof is done using metrics so there is no direct way to modify it to the non-metrizable case.

Does every Tychonoff space possess an H-compactifications of the same dimension? First we have to clarify what kind of dimension we mean. For normal spaces and covering dimension *dim* an answer can be found in Theorem 3.1.25 from [Eng95, p. 176] where it is settled that $\dim \beta X = \dim X$. But we emphasize that covering dimension is mostly defined for normal spaces only. The next proposition gives an answer in case of zero-dimensional spaces which need not to be normal.

Proposition 5.2. *Every zero-dimensional space X possess a zero-dimensional H-compactification.*

Proof. It is enough to take the biggest zero-dimensional (so called Banaschewski) compactification of X. □

Bibliography

[Bir67] Garrett Birkhoff. *Lattice theory.* Third edition. American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, R.I., 1967.

[Cha76] Richard E. Chandler. *Hausdorff compactifications.* Marcel Dekker Inc., New York, 1976. Lecture Notes in Pure and Applied Mathematics, Vol. 23.

[dGM60] J. de Groot and R. H. McDowell. Extension of mappings on metric spaces. *Fund. Math.*, 48:251–263, 1959/1960.

[Dom03] Jesús M. Domínguez. A generating family for the Freudenthal compactification of a class of rimcompact spaces. *Fund. Math.*, 178(3):203–215, 2003.

[Eng89] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[Eng95] Ryszard Engelking. *Theory of dimensions finite and infinite*, volume 10 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Lemgo, 1995.

[HNV04] Klaas Pieter Hart, Jun-iti Nagata, and Jerry E. Vaughan, editors. *Encyclopedia of general topology.* Elsevier Science Publishers B.V., Amsterdam, 2004.

[MS20] St. Mazurkiewicz and W. Sierpiński. Contribution à la topologie des ensembles dénombrables. *Fundamenta Mathematicae*, 1:17–27, 1920.

[Smi94] Yu. M. Smirnov. Can simple geometric objects be maximal compact extensions for $\mathbb{R}^n$? *Uspekhi Mat. Nauk*, 49(6):213–214, 1994.

[vD79] Eric K. van Douwen. Characterizations of $\beta\mathbb{Q}$ and $\beta\mathbb{R}$. *Arch. Math. (Basel)*, 32(4):391–393, 1979.

[vE86] Fons van Engelen. Homogeneous Borel sets. *Proc. Am. Math. Soc.*, 96:673–682, 1986.

[vM01] Jan van Mill. *The infinite-dimensional topology of function spaces*, volume 64 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 2001.